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Mathematics and Circulating Connubium

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Murngin 是澳洲土著族之一。該族的親屬組織很特殊，到目前爲止，已經討論了四十年，不但迄無定論，而且被認爲是一個引起最多爭辨，甚至最大混亂的問題。這個族曾經被調查過許多次，有非常豐富的資料，雖然一時沒有一個理論可以適合於解釋所有的現象，却仍舊引起人類學家極大的興趣去從事這項工作，包括許多如 Levi-Strauss, Leach, Murdok 等第一流的人類學家在內。

澳洲土著族的社會組織向以「分組制」(Section System) 出名，且被認爲可以用數學理論來處理。近二十年來，由於有數學家的參與，在方法上已有很大的進步。對於澳洲双交表婚的分組制的數理分析已具初步型模，但對於 Mrngin 族的母方交表婚的分析制，仍始終得不到相同的成效。

本文試圖以數學方法來分析母方交表婚的親屬結構，以爲分析 Murngin 的社會組織之用，即是：提出新的運算數或母數 (Generator) 作爲分析的基礎，並指出分組制是以分節(Segment)爲原素的集合而成的親族群，而具有代數群論中的阿伯群(Abelian group) 性質。

本文是 Kinship Mathematics and the MurnginCase (尚未出版) 一書中第四章的大部份，今先交由「思與言」發表，並請讀者先生們指正。

As is generally known, for the practice of matrilateral cross-cousin marriage in a prescriptive marriage system, three or more than three descent groups or exogamic units are necessary for the formation of the so-called 'circulating connubium' or 'marriage alliance.' But we are not well informed of the actual way in which the marriage alliance is organized when the given society possesses numerous exogamic units. Rodney Needham's analysis of the Eastern Sumba's circulating connubium presents one of the best examples (Needham 1957).

According to Needham, the marriage system of the Eastern Sumba is composed of 24 exogamic units, forming 12 marriage alliances. The number of units organizing a marriage alliance ranges from 3 to 7, the major numbers being 4, 5 and 6. Among the 24 exogamic, units over half participate in a single marriage alliance only, but nevertheless some units participate in several. The maximum number of participation is 8, with four unit belonging to this category. The other numbers of participation are 6, 5, 4 and 2, with one unit belonging to each category respectively. In other words, the Eastern Sumba's marriage alliance is not an elaborate single one in which all units participate, but an assemblage of many small alliances.

For societies genuinely regulated by unilateral descent rules, whether they are patri- or matri-lineal, the basic structuctures- totally or partially- are expressible by genealogical trees such as shown in Figures la and 2a. In these Figures the solid line represents the 'father-child link' and the dotted line the 'mother-child link.' Now the genealogical tree is divisible into two, a 'patri-tree' and a 'matri-tree,' the former representing a patrilineal descent group and the latter a matrilineal descent group. If in a given society the lineality and collaterality are ignored and members of the same generation are identified together, then the genealogical tree is simplified by the adoption of a line by which each generation is represented by two units, m (male) and f (female). We may call this simplified line a 'patri-line' or 'matri-line.' (See

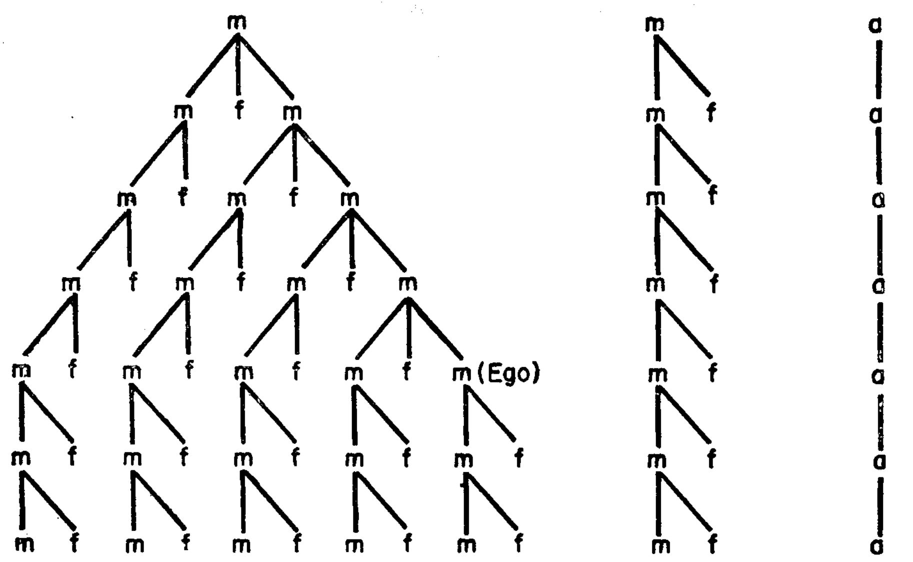


Fig. 1 Genealogical tree: patrilineal. Lines represent father-child link.
a. Patri-tree. b. Patri-line, sex-distinguished. c. Patri-line, sex-ingored.

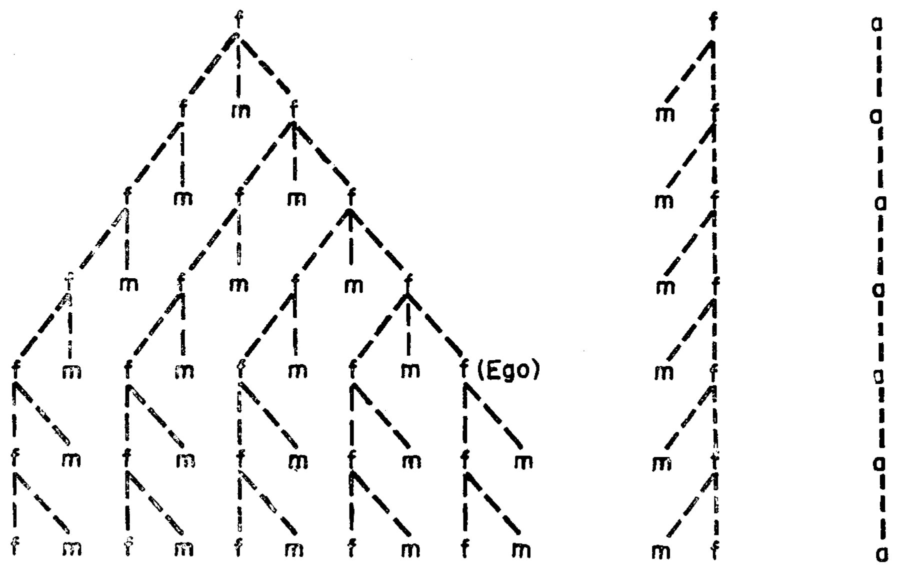
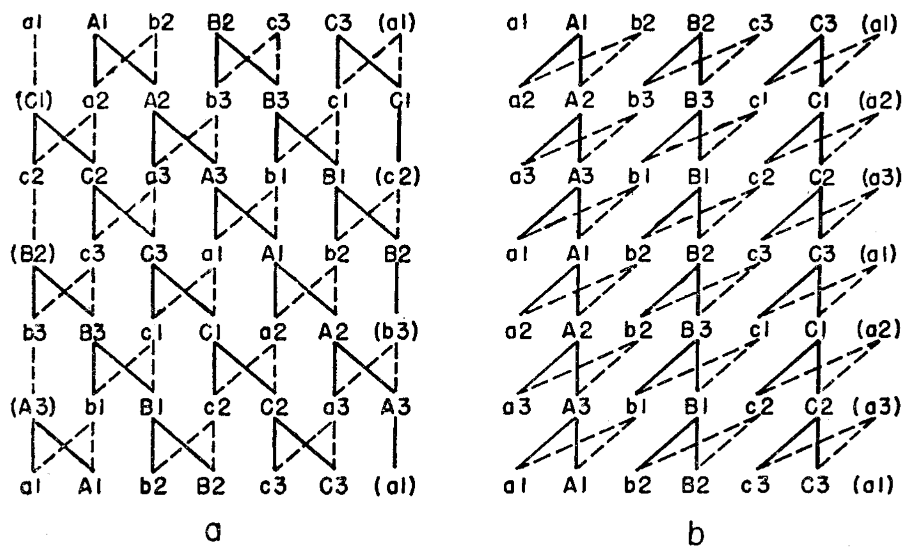


Fig. 2 Genealogical tree: matrilineal. Lines represent mother-child link.
a. Matri-tree b. Matri-line, sex-distinguished. c, Matri-line, sex-ignored.

Figures 1b and 2b.) If we ignore the sex furthermore, then each generation of the genealogical line can be expressed by a more simplified unit — sex variable 'a' (see Figures 1c and 2c). If a distinction is necessary between the two lines, we may call the former a 'sex-distinguishing patri- or matri-line' and the latter a 'sex-ignoring patri- or matri-line.' Each unit of the generation of a genealogical line is called a 'segment.'

The formation of a marriage alliance by matrilateral cross-cousin marriage is expressible by genealogical lines. The simplest one needs three descent lines. This simplest form of marriage alliance is employed for the study of the mathematical properties of the matrilateral cross cousin marriage system.

Figure 3a adopts three sex-distinguishing genealogical lines to express the simplest matrilateral cross-cousin marriage alliance, emphasizing neither patri- nor matri-lines. By tracing the solid and dotted lines, we may easily find that the marriage alliance is composed of three patri-and three matri-lines. In the given diagram patri-lines are represented by A B, C and matri-lines by 1, 2, and 3. In Eigure 3b and 3c the patri- and matrilines are emphasized respectively. If sex-ignoring genealogical line are adopted, the diagram is expressible by a more concise form. This is shown in Figure 4, where 4a represents patri-line emphasized and 4b matri-line emphasized marriage alliances. In these diagrams the descent lines are extendable limitlessly to ascending and descending generations. From these diagrams the following properties are derivable:



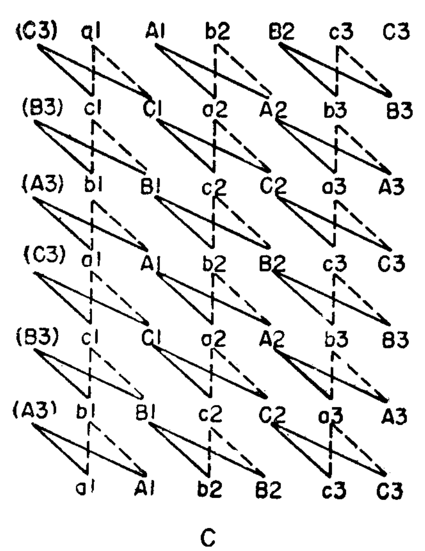


Fig. 3 Matrilateral cross-cousin marriage system composed of three descent lines.
a. Compromise form emphasizing neither patri- or matri-lines
b. Patri-lines are emphasized
c. Matri-lines are emphasized

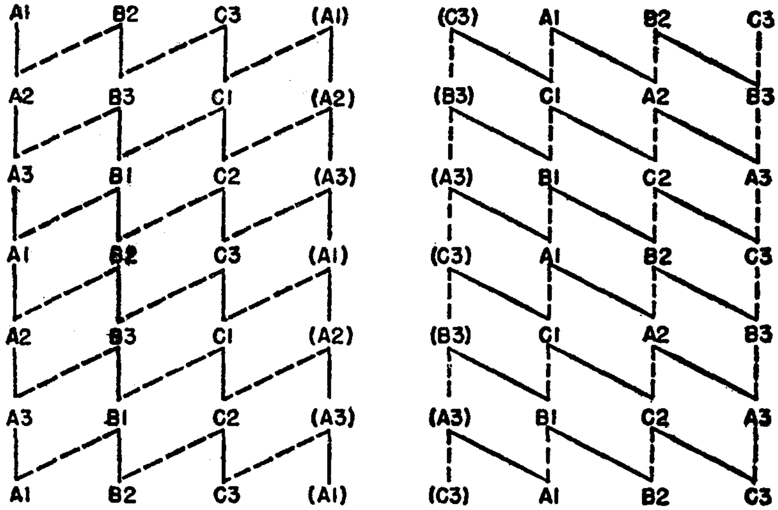


Fig. 4 Matrilateral cross-cousin marriage system composed of three descent lines in simplified form.

(1) Members of these three descent-lines associated with the same marriage alliance are categorized into the following nine segments. They are:

A1 A2 A3 B1 B2 B3 C1 C2 C3

(2) These segments are produced by the intersection of three patri-lines and three matri-lines, so the sum of the segments is equal to the product of the numbers of both descent lines. Moreover, the numbers of patri- and matri-lines are equal to the numbers of descent groups organizing the marriage alliance. Thus, if the marriage alliance is derived from n descent groups (n is 3 or larger than 3), the number of the segments Ns is deduced by the following formula:

$$Ns = n^2, \qquad n \geq 3.$$

Each patri- and matri-line consists equally of three segments and performs regular circulations. This system is regulated by 'the principle of a 3 generation cycle.' The cycle of three patri-lines is:

A1 $\longrightarrow$ A2 $\longrightarrow$ A3 $\longrightarrow$ A1
B2 $\longrightarrow$ B3 $\longrightarrow$ B1 $\longrightarrow$ B2
C3 $\longrightarrow$ C1 $\longrightarrow$ C2 $\longrightarrow$ C3

The other three matri-cycles are:

A1 $\longrightarrow$ C1 $\longrightarrow$ B1 $\longrightarrow$ A1
B2 $\longrightarrow$ A2 $\longrightarrow$ C2 $\longrightarrow$ B2
C3 $\longrightarrow$ B3 $\longrightarrow$ A3 $\longrightarrow$ C3

Perceiving the fact that each descent line is expressible by a limited number of segment (here the number is three), and that the marriage alliance consists of nine segments, the model of the system is constructable by these limited elements. This is shown by a Cayley diagram in Figure 5, where the three patri-lines are more emphasized than the matri-lines. Based on this diagram the mathematical properties of the matrilateral cross-cousin marriage system will be discussed.

In Figure 5, we adopt two generators m and f; the former represents the father-child link by a solid line, and the latter represents the mother-child link by a dotted line. The arrowhead indicates the transition of generations from ascending to descending. The number of the generation is expressed by the exponent of the generators.

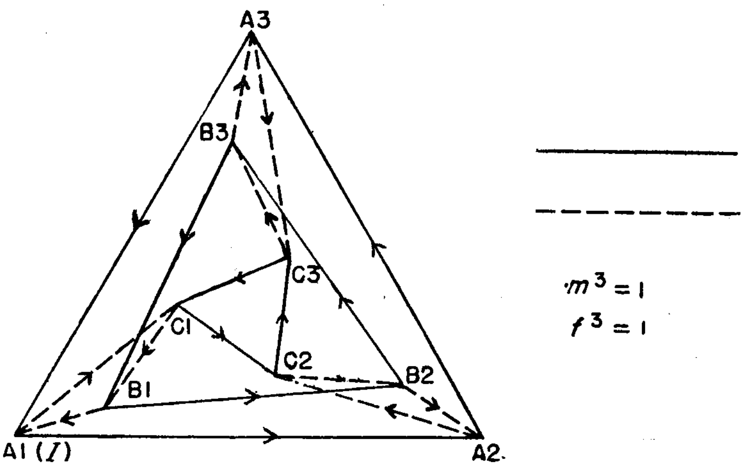


Fig. 5. Structural model of matrilateral cross-cousin marriage system composed of three descent lines. m: father-child link f: mother-child link

In this case the positive number stands for ascending generations and the negative number for the descending. For example, m^2 stands for father's father's generation; f^{-2} stands for sisert's daughter's daughter's generation. Thus the null number, m^0 or f^0, inevitably represents no ascending or descending generation, that is, Ego's generation. The exponents of the same descent-line are computable. For example,

$$m^2 \cdot m^{-1} = m^{2+(-1)} = m^{2-1} = m.$$

Suppose Ego is placed in segment A1, say, A1 = I, then starting from this point and following the patri-line in the direction of the arrowhead, the following equations can be deduced:

A1 = m^0, A2 = m^{-1}, A3 = m^{-2}, A1 = m^{-3}, A2 = m^{-4}, A3 = m^{-5}, ······

In this case every descent line is regulated by the principle of a 3-generation cycle, that means the exponent is determined by modulo 3 or $m^{-3} = I$. By the application of this rule the above-listed equations can be rewritten as follows:

A1 = I, A2 = m^{-1}, A3 = m^{-2}, A1 = I, A2 = m^{-1}, A3 = m^{-2}, ······

Now we convert the transition in the direction of the ascending generation, the ruling principle becomes $m^3 = I$, and each segment can be expressed as follows:

A1 = I, A3 = m, A2 = m^2, A1 = I, A3 = m, A2 = m^2, ······

Synthesizing the above two cases, the ruilng principle is expessible by $m^{\pm 3} = I$ and each segment is tabulated as follows:

A1 = I A2 = $m^{\pm 3-1}$ A3 = $m^{\pm 3-2}$

For the same reason, the matri-line passing through A1 is regulated by the principle of $f^{\pm 3} = I$, and its segments are expressible by the following equations:

A1 = I B1 = $f^{\pm 3-2}$ C1 = $f^{\pm 3-1}$

The paths to the other segments are numerous, for example, the approach to C3 is:

$$C3 = f^2m, \quad mf^2, \quad f^{-1}m^{-2}, \quad m^{-2}f^{-1}, \cdots\cdots$$

Among then the form $f^{\pm 3-n}m^{\pm 3-n}$ is chosen as a representative for each segment. The remaining four segment are expressed as follows:

$$B2 = f^{\pm 3-2}m^{\pm 3-1} \qquad B3 = f^{\pm 3-2}m^{\pm 3-2}$$

$$C2 = f^{\pm 3-1}m^{\pm 3-1} \qquad C3 = f^{\pm 2-1}m^{\pm 3-2}$$

Conbining the above-mentioned equations, we find that each segment is replacible by the multiplication of two other segments. For example, if $B2 = f^{\pm 3-2}\, m^{\pm 3-1}$ is replaced by $B1 = f^{\pm 3-2}$ and $A2 = m^{\pm 3-1}$, the following equation can be deduced:

$$B2 = B1 \cdot A2$$

Thus the multlplication of any two segments will result in one of the given segments. The product of the multiplication of these nine segments is shown as follows:

Tabe 1. 9-segment multiplication table.

•	A1	A2	A3	B1	B2	B3	C1	C2	C3
A1	A1	A2	A3	B1	B2	B3	C1	G2	C3
A2	A2	A3	A1	B2	B3	B1	C2	C3	C1
A3	A3	A1	A2	B3	B1	B2	C3	C1	C2
B1	B1	B2	B3	C1	C2	C3	A1	A2	A3
B2	B2	B3	B1	C2	C3	C1	A2	A3	A1
B3	B3	B1	B2	C3	C1	C2	A3	A1	A2
C1	C1	C2	C3	A1	A2	A3	B1	B2	B3
C2	C2	C3	C1	A2	A3	A1	B2	B3	B1
C3	C3	C1	C2	A3	A1	A2	B3	B1	B2

The table reveals that:

(1) The marriage alliance of the matrilateral cross-cousin marriage system consisting of three descent lines (G^3_3: superscripts indicate the generation cycle and subscripts indicate the number of descent groups) is expressed by a set of nine segments as its elements as shown in the following diagram.

$$G^3_3 = \left\{ \begin{array}{ccc} A1 & B1 & C1 \\ A2 & B2 & C2 \\ A3 & B3 & C3 \end{array} \right\} \qquad \begin{array}{l} m\text{: mod} = 3 \\ f\text{: mod} = 3 \end{array}$$

(2) 'Generation transition' is a binary operation with the elements.

(3) The set contains I (A1 in this case) as an element and, for any element u, $u \cdot I = I \cdot u = u$. I is the identity element or unit element.

(4) Since I occurs precisely once in each row and column, the axiom in inverse is satisfied. We can determine at once the inverse of any group element from the table. For example, the configuration

$$\begin{array}{ccc} & & C3 \\ & & \vdots \\ B2 & \cdots\cdots\cdots\cdots\cdots\cdots & A1 \end{array}$$

shows us that $(B2)^{-1} = C3$ or $(C3)^{-1} = B2$.

(5) The group is commutative. For example; $B2 \cdot C2 = C2 \cdot B2$.

(6) The rows and columns are permutations or rearrangements of the elements in the top row and left column, respectively–the 'coincidence' previously observed.

(7) Associatively; This property is satisfied as

$$(A2 \cdot B3) \cdot C1 = A2 \cdot (B3 \cdot C1) = A1$$

In dealing with group elements and their relations, it is sometimes necessary to be able to answer this question: If a and b are known elements of a group, is there an element x of the group such that $ax = b$? We claim that $x = a^{-1}b$ is the group element we seek, for

$$a(a^{-1}b) = (aa^{-1})b = Ib = b$$

that is, $x = a^{-1}b$ satisfies the group equation $ax = b$. Applying this to kinship problem, for such question as $C2 \cdot x = B1$ (What are my B1 to my C2?) The answer is

$$x = (C2)^{-1} \cdot B1 = B3 \cdot B1 = C3$$

Utilizing the multiplication table, we also can find the answer immediately. The configuration

$$\begin{array}{ccc} & & C3 \\ & & \vdots \\ C2 & \cdots\cdots\cdots\cdots\cdots\cdots & B1 \end{array}$$

shows us that $C2 \cdot C3 = B1$, C3 is the answer.

In the above description, we have discussed the mathematical properties of the marriage alliance of a matrilateral cross-cousin marriage syste m which is organized by descent groups. We see that the nine elements are performing a high mathematical function. But, owing to the adoption of a non-numerical system for their symbols, they are prevented from a direct mathematical manipulation. No binary operation is possible without the help of a multiplication table. It would be ideal if a system of a set of numerical notations were explored which fits the operation.

For this purpose, the exponents of generators are most suitable resources for the construction of a numerical notation system. We have chosen the form $f^{\pm 3-n}\ m^{\pm 3-n}$ as a representative for each segment. Now we adopt a two place system and assign the first place to the n value of the exponent of f and the second place to the n value of the exponent of m. Thus, for example, $C3 = f^{\pm 3-1}\ m^{\pm 3-2}$ is represented as 12. But this is hardly distinguishable from the numerical notation system for the 'kinship category' (Harvey and Liu 1967). So in this case S is added in front for discrimination. C3 is represented by S12 instead of 12. Now group G_3^3 can be expressed as a set of 9 elements in the following diagram;

$$G_3^3 = \left\{ \begin{array}{lll} S00\ (A1) & S10\ (C1) & S20\ (B1) \\ S01\ (A2) & S11\ (C2) & S21\ (B2) \\ S02\ (A3) & S12\ (C3) & S22\ (B3) \end{array} \right\} \begin{array}{l} m;\ \mathrm{mod} = 3 \\ \\ f;\ \mathrm{mod} = 3 \end{array}$$

The mathematical properties of G_3^3 are to be treated as follows:

(1) Binary operation; The generations of the same descent line (or the exponents oi the same generators) are computable. The formula is

$$Sx_1y_1 \cdot Sx_2y_2 = S(x_1 + x_2)\ (y_1 + y_2)$$

For example, $S10 \cdot S21 = S(1 + 2)\ (0 + 1) = S01$

(2) Identity; S00 is the identity segment. For example

S00 · S21 =S21 · S00 = S21

(3) Associativity; This property is satisfied as

(S21 · S11) · S02 = S21 · (S11 · S02) = S01

(4) Inverses; Each element (or segment) has its inverse in the set.

The formula is

$(Sxy)^{-1} = S(3 - x)\ (3 - y)$

For example

$(S21)^{-1} = S(3 - 2)\ (3 - 1) = S12$

(5) Commutative: This property is shown as

S12 · S02 = S02 · S12 = S11

Because it is commutative and because of the finite number of its elements, this group is called a 'finite Abelian group'. The multiplication table is rearranged as follows:

Table 2. Muptlication table in numerical notation system

·	S00	S01	S02	S10	S11	S12	S20	S21	S22
S00	S00	S01	S02	S10	S11	S12	S20	S21	S22
S01	S01	S02	S00	S11	S12	S10	S21	S22	S20
S02	S02	S00	S01	S12	S10	S11	S22	S20	S21
S10	S10	S11	S12	S20	S21	S22	S00	S01	S02
S11	S11	S12	S10	S21	S22	S20	S01	S02	S00
S12	S12	S10	S11	S22	S20	S21	S02	S00	S01
S20	S20	S21	S22	S00	S01	S02	S10	S11	S12
S21	S21	S22	S20	S01	S02	S00	S11	S12	S10
S22	S22	S20	S21	S02	S00	S01	S12	S10	S11

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